

Focus in High School Mathematics: Reasoning and Sense Making

Christine Thomas

Fred Dillon

NCTM Board of Directors



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

Pat is in a class of 31 students. The teacher asks, “Pick a group of three other students to work with you on a presentation for the class.”

From the 30 remaining students, in how many different ways can Pat select a group of three?



Focus in High School Mathematics

In a traditional class, the students may say,

“There is a formula that tells us that. But, I’m not sure what it is.”

Or, the teacher may say,

“Order doesn’t matter, so we use the combinations formula.”



Focus in High School Mathematics

Typically, most texts simply give a problem and move directly to a formula. Students who have studied formulas with little context have a difficult time recalling them.



Focus in High School Mathematics

If students have learned counting formulas and understand them, this is a question of reasonably deciding which formula to use.

What kind of questions would you want students to be asking?



Focus in High School Mathematics

Students may look at this as simply a counting problem when beginning.

For example, Joe may say,
“There are 30 ways to pick the first student, then 29 ways to pick the next, and only 28 ways to pick the third student.”



Focus in High School Mathematics

Then Joetta may say,
“That means that if Pat picks student Ann, Ben and Carl it is different than picking Carl, Ben and Ann. But, it is the same three people. We need to take that into consideration.”



Focus in High School Mathematics

At this point, the discussion turns to dividing out different orders of the three objects and ways to count those orders.

Generalizing the discussion develops not only a formula for counting combinations, but for counting permutations.



Focus in High School Mathematics

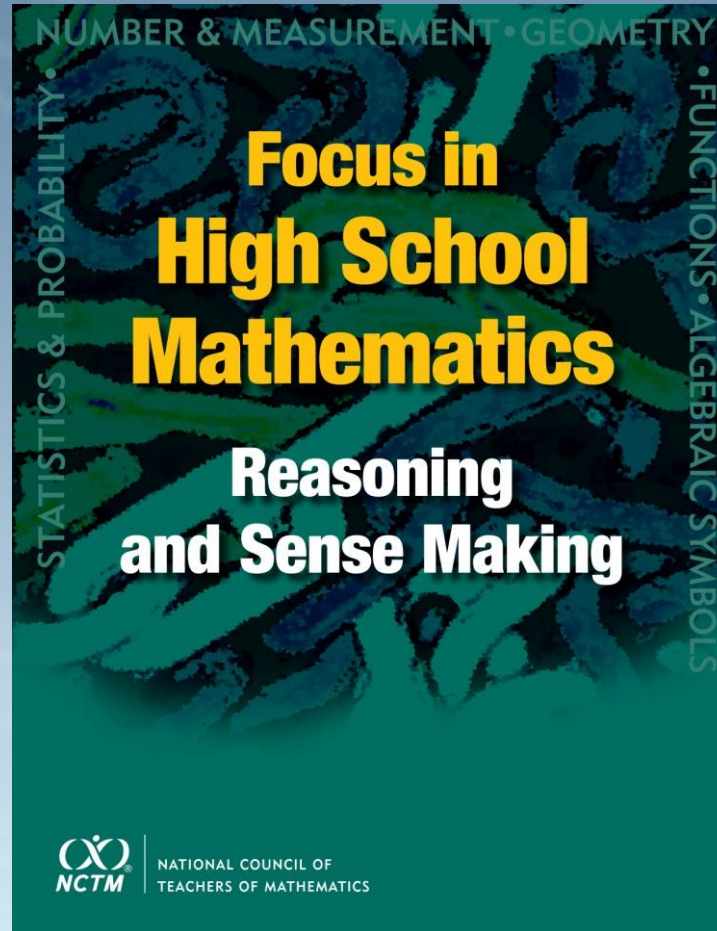
Since the students have reasoned through the steps to create the formulas from a problem context, their understanding is better and the likelihood of remembering at least the procedure to be able to count in similar situations is in place.

See also “A Welcome Focus” *Mathematics Teacher*, October 2009, pp.168-171.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Released in 2009



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

“This publication advocates that all high school mathematics programs focus on reasoning and sense making.”

What *IS* reasoning and sense making?



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

Focus in High School Mathematics is NCTM's response to the call for a 9-12 document after the release of *Curriculum Focal Points*.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

This document is NOT a list of topics to be covered. There are many such lists at the state level, through Achieve and many other places.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

This document was created to be used as the conceptual framework to guide the development of future publications and tools related to 9-12 mathematics curriculum and instruction.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Focus in High School Mathematics

This is a way to approach instruction no matter what content you are teaching.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Task Force Members

- **Ruth Casey**, Chair and Board Liaison
University of Kentucky
- **John A. Carter**, NCTM Board Liaison
Adlai E. Stevenson (Illinois)
High School
- **Martha Aliaga**
American Statistical Association
- **Hyman Bass**
University of Michigan
- **Kaye Forgione**
Achieve, Inc.
- **Albert Goetz**, Staff Liaison
NCTM
- **W. Gary Martin**
Auburn University
- **William McCallum**
University of Arizona
- **Zalman Usiskin**
University of Chicago



Writing Group Members

- **John Carter**, Board Liaison
Adlai E. Stevenson High School
- **Susan Forster**
Bismarck High School
- **Roger Howe**
Yale University
- **Gary Kader**
Appalachian State University
- **Henry Kepner**
University of Wisconsin-Milwaukee
- **W. Gary Martin**, Chair
Auburn University
- **William McCallum**
University of Arizona
- **Eric Robinson**
Ithaca College
- **Judith Quander**, Staff Liaison
NCTM
- **Vincent Snipes**
Winston-Salem State University
- **Patricia Valdez**
Pajaro Valley High School



Reasoning and Sense Making

Reasoning habits are productive ways of thinking that become customary in the process of mathematical inquiry and sense making. Reasoning habits include:

- Informal explanation
- Inductive observations
- Justification
- Formal deduction

Reasoning and Sense Making

Reasoning and sense making is an evolution of NCTM's long-standing position that problem solving should be the emphasis of mathematics teaching and learning.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Reasoning and Sense Making

“The processes of mathematics – Problem Solving, Reasoning and Proof, Connections, Communication, and Representation - are all manifestations of the act of making sense and of reasoning ...” (p. 5)



Reasoning and sense making should occur every day in the high school mathematics classroom.

Reasoning Habits

Developing Reasoning Habits

- Provide tasks
- Good questioning techniques
- Provide adequate wait time
- Resist the urge to tell students everything
- Expect students to communicate reasoning

See page 9 for the complete list



Reasoning and Sense Making

Reasoning is the foundation of mathematical competence

- Conceptual understanding
- Procedural fluency
- Strategic competence
- Adaptive reasoning
- Productive disposition

See page 12 for details



Reasoning and Sense Making

Statistical Reasoning

Students need to make sense of data with the capacity to reason with statistics.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Reasoning and Sense Making

Mathematical Modeling

“The connections between mathematics and real-world problems developed in mathematical modeling add value to, and provide incentive and context for, studying mathematical topics.” (p. 13)



Reasoning and Sense Making

Technology

- Technology can be used to advance the goals of reasoning and sense making
- Technological tools can relieve students of burdensome calculations and can facilitate the search for patterns and relationships and the formation of conjectures.



Reasoning and Sense Making

“Reasoning and sense making are integral to the experiences of all students across all areas of the high school mathematics curriculum.”

p. 17



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Reasoning and Sense Making

Focus in High School Mathematics highlights reasoning opportunities in five specific content areas:

- Number and Measurement
- Algebraic Symbols
- Functions
- Geometry
- Statistics and Probability



Reasoning and Sense Making

Each example includes a discussion of the key mathematical elements addressed in the task and reasoning habits required of students.



Using Tasks

Fuel for Thought

A teacher gives her students the following prompt taken from an article in the *New York Times* (Chang 2008) and asks them to explain their reasoning.



Fuel for Thought

Quiz time: Which of the following would save more fuel?

- a) Replacing a compact car that gets 34 miles per gallon (MPG) with a hybrid that gets 54 MPG
- b) Replacing a sport utility (SUV) that gets 18 MPG with a sedan that gets 28 MPG.
- c Both changes save the same amount of fuel.



Fuel for Thought

Discuss with a small group possible solutions for this problem.

Think about different ways a student may approach this problem.



Fuel for Thought

Sample response:

I see that the change from 34 to 54 MPG is an increase of 20 MPG, but the 18 to 28 MPG change is only a change of 10 MPG. So, replacing the compact car saves more fuel.



Fuel for Thought

Sample response:

The change from 34 MPG to 59 MPG is an increase of about 59% while the change from 18 to 28 MPG is an increase of only 56%. So the compact car is a better choice.



Fuel for Thought

Sample response:

I thought about how much gas it would take to make a 100-mile trip.

Compact car:

$100 \text{ miles}/54\text{MPG} = 1.85 \text{ gallons used}$

$100 \text{ miles}/34\text{MPG} = 2.94 \text{ gallons used}$

SUV:

$100 \text{ miles}/28\text{MPG} = 3.57 \text{ gallons used}$

$100 \text{ miles}/18\text{MPG} = 5.56 \text{ gallons used}$



Fuel for Thought

The compact car saved 1.09 gallons while the SUV saved 1.99 gallons for every 100 miles. That means you actually save more gasoline by replacing the SUV.



Discourse

Who is correct?

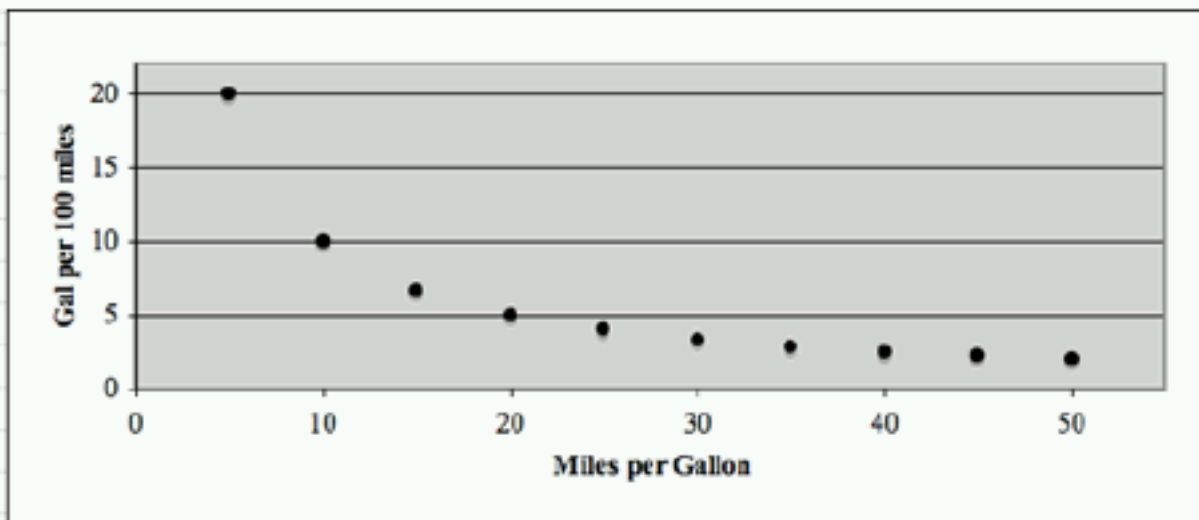
Defend your answer!



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Fuel for Thought

MPG	Gal per 100 miles
5	20
10	10
15	6.66666667
20	5
25	4
30	3.33333333
35	2.85714286
40	2.5
45	2.22222222
50	2



Using Tasks

Fuel for Thought

Key Mathematical Elements

- Number and Measurement (reasonableness of answers and measurements)
- Functions (multiple representations of functions)

Reasoning Habits

- Analyzing and problem (seeking patterns and relationships)
- Reflecting on a solution (interpreting a solution; reconciling different approaches; refining arguments)



Strategy: Multiple Entry Points

Use problems that have multiple entry points so that students at different levels of mathematical experience and with different interests can all engage meaningfully in reasoning about a problem.



Using Tasks – again!

Find at least one solution containing only positive integers for the equation:

$$28x + 30y + 31z = 365$$



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Mathematics League, INC., question 4, October, 2009 www.themathleague.com
Problem cannot be reproduced without permission from The Math League

Using Tasks – again!

One student's solution:

Because 30 ends in zero, the y -term cannot be used to create the 5 in the ones place.

If I let $x = 1$, then the number for z has to be something that ends in 7.

I tried $x = 1$, $z = 7$ and found $y = 4$.



Using Tasks – again!

Are there other solutions?



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Using Tasks – a different student

I think 365 represents the days in the year.

Then 28, 30 and 31 are possible month lengths.

So, one month has only 28 days, 4 have only 30 and 7 have 31 days.

How can you get started?

Recast the material as questions.
The opening counting example was a text problem that was remade into a reasoning and sense making task.



How can you get started?

- Provide opportunities for students to muck around with the task
- Believe that students can investigate and derive relationships and mathematical concepts

How can you get started?

Finding x- and y-intercepts
Students are given rules

How could you change this so that students are challenged to reason about the x- and y- intercepts?



How can you get started?

Finding x- and y-intercepts

- Provide graphs or tables and students identify characteristics of the intercepts
- Ask students why the textbook rules work
- Provide a “messy” equation that’s difficult to graph: $33x + 14y = 231$



Tips for Developing Reasoning Habits

- Provide tasks that require students to figure things out for themselves
- Ask students to restate problems in their own words
- Give students time to analyze a problem intuitively, explore the problem further using models, and then proceed to a more formal approach
- Resist the urge to tell students how to solve problems
- Ask questions that will press students thinking

National Council of Teachers of Mathematics (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston, VA



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Tips for Developing Reasoning Habits

- Provide adequate wait time
- Encourage students to ask probing questions of themselves and each other
- Expect students to communicate their reasoning to their classmates and teacher verbally and in writing
- Highlight exemplary examples and have students reflect on what makes them effective
- Establish a classroom climate in which students feel comfortable sharing their mathematical arguments and critiquing the arguments of others

National Council of Teachers of Mathematics (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston, VA



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

A new study has found people who drink one or more soft drinks a day have close to a 50% higher risk of developing the heart disease precursor metabolic syndrome than people who drink less than one soda a day - and it doesn't matter if it is regular soda or a diet soda.

<http://www.cnn.com/2007/HEALTH/conditions/07/24/diet.sodas/index.html>



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

Metabolic syndrome is a set of health problems – including high waist circumference, high blood pressure, low levels of “good” cholesterol – that have been strongly linked to developing heart disease, stroke, and diabetes.

<http://www.cnn.com/2007/HEALTH/conditions/07/24/diet.sodas/index.html>



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

The study, in the American Heart Association journal *Circulation*, looked at more than 6,000 healthy people, who showed no signs of metabolic syndrome, and then followed up. After four years, 53% of people who drank an average of one or more soft drinks per day developed metabolic syndrome. Those who drank one or more diet soft drinks a day were at a 44% higher risk.

<http://www.cnn.com/2007/HEALTH/conditions/07/24/diet.sodas/index.html>



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

In conjunction with the study, AOL maintained an ongoing poll.

Responses to the Question, "Will You Stop Drinking Diet Soda Because of This Study?"

Response	Frequency	Percent
No	15,718	52%
I don't drink diet soda	8,224	27%
Yes	6,531	21%
Total votes	30,473	100%

Results of poll after 3 days

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

After reading the two media clips and the results of the AOL poll, start recording items in two lists, one headed “I Notice That...” and the other headed “I Wonder About..”

Jot down any “notices” and/or “wonders” that strike you about the information presented, the way in which the study was conducted, and the conclusions from the study.

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

I notice that....

I wonder about....



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

I notice that...

- Regular soft drinks were worse than diet drinks.
- Nobody had the syndrome before the study started.
- A lot of people said they would stop drinking soda because of this study.
- Only 21% of people in the poll said they would stop drinking diet soda.
- These clips don't tell us what kind of soda.
- It isn't much difference between 53% and 44%, and both are about half.
- About half the soda drinkers didn't develop the syndrome.
- The study didn't report any people who drank both diet and regular sodas.
- The study makes you feel that it's dangerous to drink soda.
- Over 30,000 people responded to the poll; that's a lot.
- The American Beverage Association seems to be saying that more than one diet drink a day is "over-consuming."

I wonder about...

- What were the percentages of the syndrome for people who drank less than one soft drink a day after four years?
- What does "44% more likely" mean? Is that like the 53%, meaning 44% of the diet drinkers had the syndrome?
- Where did these 6000 people come from?
- Is it a random sample of 6000, and if so, drawn from what group of people?
- What else were these people eating or drinking?
- Why did the poll just ask about diet soda; why not regular, too?
- How reliable is people's reporting of their soft drink consumption—would they remember? Did the study ask people to keep track?
- Who are the people who responded to the poll? Do we know anything about them?
- Is the American Heart Association saying that soft drinks cause these diseases?



Soft Drinks and Heart Disease

After analyzing the data, what conclusions can you draw? Would you stop drinking soft drinks on the basis of this study? Why or why not?



Soft Drinks and Heart Disease

Sample student group response:

This study presents strong evidence that long-term use of soft drinks leads to increased risk of heart disease. The results show a 53% increase in risk for regular pop and a 44% increase in risk for diet pop. It is like cigarette smoking – even when people know there are risks, they don't think the risks apply to themselves. So, the majority of the people in the poll (even though it is voluntary response and not reliable) are not willing to change.

Students' response, Strongsville, OH Integrated College Mathematics, grade 12, October 2009



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

Sample student group response:

There are a lot of questions raised by this study that we need to have answered. How were the 6,000 people selected? Was the soft-drink consumption self-reported? Are there other factors that were not studied, such as, do people who drink more pop have other bad habits - eating more overall or eating a “bad” diet? We cannot agree with the study when so many details are left out.

Students' response, Strongsville, OH Integrated College Mathematics, grade 12, October 2009



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Soft Drinks and Heart Disease

What makes a good study?



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

What makes a good study?

- Indicate how participants were selected
- Provide background on participants
- Carefully explain the meaning of numbers and percentages reported
- Be open and honest about other potential variables that could have influenced the results of the study

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

What makes a good study?

- Explain the mechanisms for any polling that was done and describe the people who would likely respond
- Suggest what the next steps should be in the research

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Habits of Mind

Looking for patterns and relationships
by –

- analyzing and explaining variation
- making preliminary deductions and conjectures

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Habits of Mind

Selecting representations or procedures by –

- choosing and critiquing data collection strategies based on the question
- considering the random mechanisms behind the data
- drawing conclusions beyond the data

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Habits of Mind

Evaluating a chosen strategy by –

- Questioning whether the observations make sense within the problem context
- evaluating the consistency of different components of the analysis

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Habits of Mind

Connecting different representations by—

- understanding the sensitivity of an analysis to various components
- connecting conclusions and interpretations to the context

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Habits of Mind

Checking the reasonableness of an answer by—

- considering and evaluating alternative explanations
- understanding the allowable scope of conclusions
- determining whether a conclusion based on the data is plausible

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Equity

- High expectations for all students
- Courses offer rich opportunities for reasoning and sense making
- Students see mathematics as important for their lives and future careers



Now It's Your Turn

What about promoting reasoning and sense making in teaching your required curriculum?

Let's examine how we might do this using a content standard found in a state's standards and in the common core standards.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Standard – High School Algebra

State Standard (High School Algebra)

Students will simplify and operate with radical expressions, polynomials, and rational expressions.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Standard – High School Algebra

Factor expressions by greatest common factor, grouping, trial and error, and special products limited to the formulas below.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)(x - y) = x^2 - y^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$



Common Core Standard – High School Algebra

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
2. Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*



Now It's Your Turn

Talk at your table.

How might you develop a lesson that promotes reasoning and sense making?

$$x^2 - y^2 = ?$$



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Now It's Your Turn

Talk at your table.

What are the key elements?

What are the reasoning habits?

How will you engage your students?

How can you extend this to $x^4 - y^4 = ?$



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Now It's Your Turn

Talk at your table.

What prior knowledge is required of students?

What questions will you ask?

How will you engage your students?



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

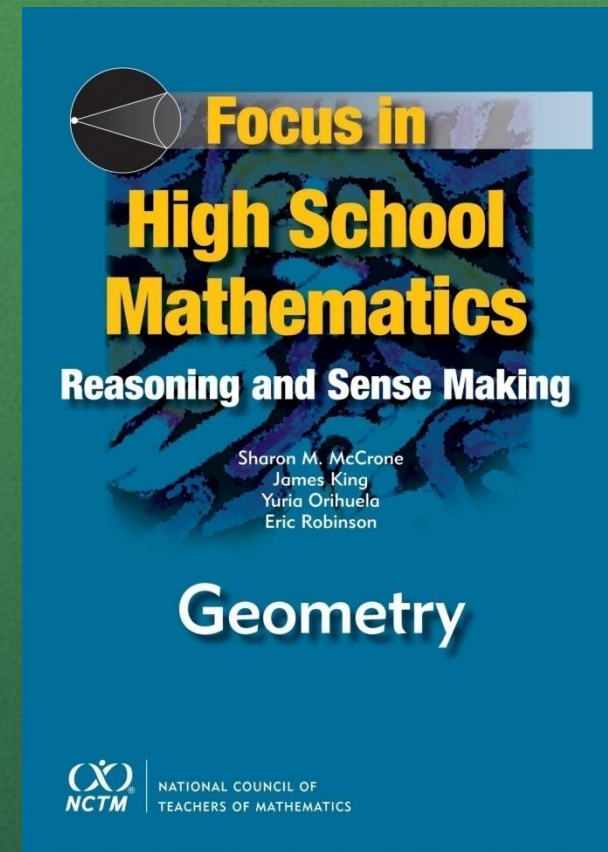
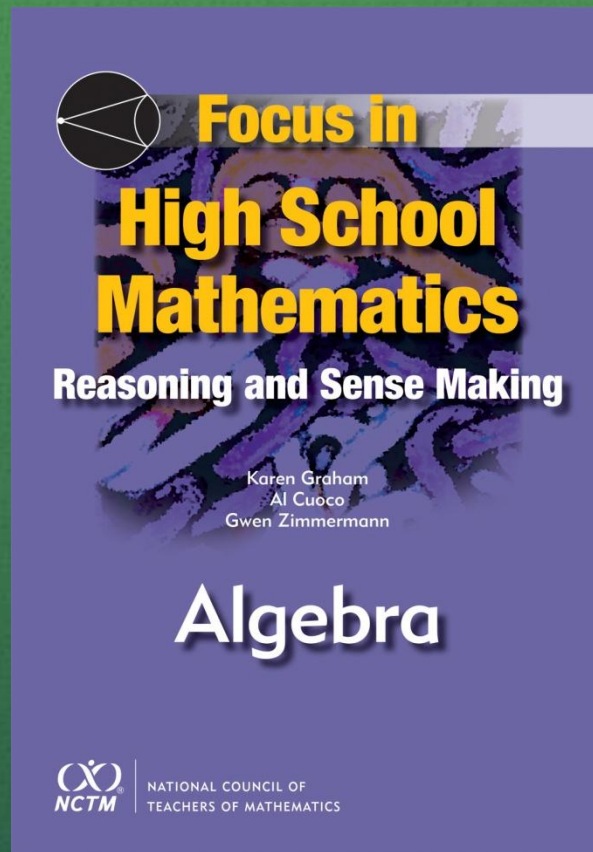
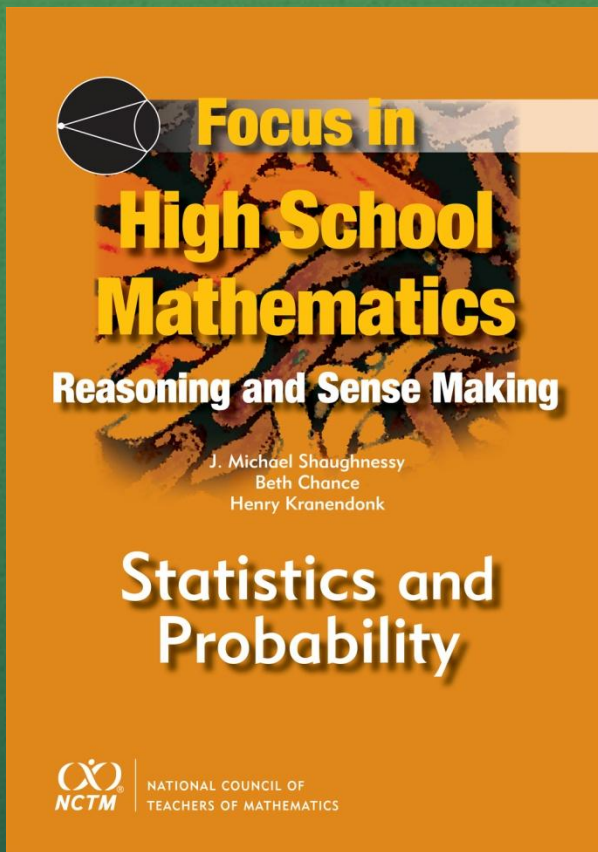
Reasoning and Sense Making

Coherence

Components of the educational system must work together for the good of all students

- Curriculum
- Assessment
- Instruction





These will be followed by books that offer examples of ways to apply reasoning and sense making while working to achieve equity, and to incorporate technology to promote reasoning and sense making.

You Can Help!

When you use reasoning and sense making in your classroom, consider writing an article for an NCTM journal to share with your colleagues.

- My Favorite Lesson
- Department
- Feature Article
- Reader Reflection
- Sound Off
- Focus Issue



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

To Receive a Copy of this presentation

Request a PDF of this talk by e-mailing:

dillon314@sbcglobal.net

or

cthomas11@gsu.edu



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Reasoning and Sense Making

For more information:

<http://www.nctm.org/hsfocus>



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Thanks!

Thank you for attending!



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS